Inter-Dependence of Capital in Input-Output Framework

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THE conventional form of the Leontief model deals with current production. It involves *(i)* a matrix of transactions representing the flows of the commodities between the different industrial sectors engaged in current production; *(ii)* a vector of final demands placed on the producing sectors by the non-producing sector, and *(ii)* a matrix of coefficients relating the flows in the sectors with the total transactions of the inter-industrial blocks. Representing the transaction matrix in partitioned form we can write :

$$\left[\begin{array}{c|c} x & f \\ \hline y^1 & 0 \end{array}\right]$$

where x is a sub-matrix of order n-I, f a vector of final demands, y^1 a row vector representing net factor costs and 0 a scalar. If X be a vector of total outputs of n-1 industries the accounting identities may be written in matrix notation as

$$X = x \quad i + f \qquad (1)$$

where f is a column vector with n-I elements and x is the matrix of intermediate outputs and i the unit vector. The assumption that inputs are related to outputs, can be expressed as

$$x = a\ X \qquad (2)$$

where a, the coefficient matrix is of order n-l and X is a diagonal matrix whose elements are the elements of X

By substitution, we obtain

$$X = (I - a)^{-1} \quad f \qquad (8)$$

where I is a unit matrix of order n-I. This gives us the socalled open model of Leontief connecting the total output vector X with the final output vector f by means of the matrix multiplier.

The matrix multiplier $(i - a)^{-1}$ shows in each column the supplies directly and indirectly required from every industry to satisfy a unit of final demand for the product of the industry named at the top of the column.

The matrix multiplier throws light on the structure of industry by showing the indirect relationship existing between different branches of production.

As may be seen in the above description of the Leontief model the use of the model is mainly restricted to current production levels and its structure.

In many countries, under different levels of planning, however quick description and prediction is required of capital implications and inter-sectoral capital relationships of given final demand targets. That is, planners are interested in a simple concept relating the final demands with capital capacities directly and indirectly required, as is given in the case of the matrix multiplier which provides a simple expression of the direct and indirect current outputs associated with given levels of final demands. In such cases the Leontief model is used initially to find out the current output levels associated with a unit of final demand and subsequently this is translated into capital through the use of some familiar device like the *output-capital* ratios defined by

$$X = \hat{\lambda} K \qquad (4)$$

where X is a vector of outputs, K a vector of capital employed in sectors arid X, a diagonal matrix relating capital to output. In this case capital is defined as a composite commodity evaluated at some base period price, assuming there is no change in its composition of the different components entering into it during the period under consideration.

It is suggested here that instead of using these two models in sequence we may develop a matrix describing capital relationship between different sectors by combining these two steps into a single step defined by a single coefficient. Thus if x_{ij} is the flow from i to j and if, we have,

$$x_{ij} = a_{ij}\ X_j$$

defining the level of input of type i to output of type j we can multiply into this relationship from (4) giving

$$X_j = \lambda_j,\ K_j$$

so that we have,

$$x_{ij} = a_{ij}\ \lambda_j\ K_j = \beta_{ij},\ K_j \ \text{(say)}$$

β_{ij} then gives us the amount of the ith commodity directly required to keep a unit of capital of the jth type in activity. Assuming therefore, that there exists a vector of capital output ratios as in (4) we can substitute this into (3) to obtain

$$X = \hat{\lambda} K = (1 - a)^{-} f,$$

$$\text{or, } K = \hat{\lambda}^{-1} (1 - a)^{-1} f = \beta^{-1} f,$$

where β^{-1} is the matrix obtained by matrix multiplication from $\hat{\lambda}^{-1} (1 - a)^{-1}$. The elements of β^{-1} define the direct and indirect capital requirements of a specified level of final demand.

The advantage of integrating the capital output ratios into the matrix instead of considering them in sequence is that the inverse matrix β^{-1} now gives the sectoral capital repercussions of final demands directly and indirectly involved in a single expession.

While this approach is not probably of any new theoretical implications it may have use in practical operations where capital implications of different types of programmes may have to be compared.

We may demonstrate by a simple illustration the practical use of this approach. Let the input coefficient matrix (1— *a)* be denoted by the following:

1	— 0·6	— ·2
— 0·1	1	— 0·2
— 0·5	— 0·2	1

Further, let output-capital ratios be denoted by the following diagonal matrix.

2		
	3	
		3

Then the matrix multiplier $(1 - a)^{-1}$ is given by the following matrix :

1·3043	·8696	·4348
·2717	1·2228	·2989
·7065	·6793	1·2772

and matrix β^{-1} is denoted by

2·6086	1·7392	·8696
·8151	3·6684	·8967
2·1195	2·0379	3·8376

We are already familiar with the use of the matrix multiplier which tells us in effect that for a unit of final demand of type 2 the direct and indirect consumption of output of sector 1 in sector 2 is -8696. If we now turn to the second matrix denoted by β^{-1} we can say that the direct and indirect capital requirement of type 1 to produce a unit of final demand of a commodity of type 2 is 1-71302. If we are interested for example in the question what is the capital implication of type 1, 2, and 3 in a bill of goods of unit 1, of type 1 respectively, this is given by the vector 2-6086, -8151, and 2-11 !)5.

The matrix β^{-1} thus gives us a convenient way of summarising the capital implications of any bill of goods just as the matrix multiplier gives us the total output implications of a given bill of goods.

II

To pass from this static study of the dynamic form of the Leontief model, we first expand the final demand into its components, e.g. consumption and capital goods supplied by the different sectors in the accounting identity given in (1). Expanding (1) we can write

$$X = x \cdot i + C + I(s)$$

where I(s) denotes the vector of supply of capital goods from different sectors.

Let us now define a matrix of capital input ratios which gives the composition of a unit of capital used in the different sectors. Let this matrix be denoted by B. Since *I (s)* gives the vector of capital goods supplied, breaking it up into the components with the help of the matrix B, we can write

$$I(S) = B \cdot I(d)$$

where *I(d)* is the vector of investments in different sectors.

With the above two sets of equations, we can now write

$$(1 - a) \lambda K(t) = C + B.I(d) = C + B[K(t) - K(t-1)]$$

This gives for assigned consumption levels the dynamic form for the growth of capital in all sectors.

III

The present model can be usefully extended into the field of interregional planning studies. Its use is specially assured by the fact that in most planning economies capital shortage and the problem of regional location of specific types of capital make it necessary to study alternative ways of fixing the targets so that the capital existing in different regions are best utilised in course of producing the given final output for the nations. We shall, in the present case, first formulate a conventional type of regional model which combines input-output with some form of linear programming and then show how by translating it into the terminology of capital we can make more significant use of it for developmental purposes. Let the balance equations for the *nth* region be written in the form

$$\sum_j {}_n a_{ij} X_j + \sum_m {}_{mn}X_i + {}_nC_i + {}_nI(s)_i = {}_nX_i$$

where the 1st subscript refers to the region of origin and the second subscript to that of destination, other symbols being interpreted for the same variables as before but as sealars.

The constraints on the transport available from region to region may be expressed by the following equations :

$$\sum_m \sum_j W_j \cdot {}_{mn}X_j \leqq T_{mn}$$

where Wj denotes the weight per Leontief unit of a commodity of a. particular jth type and T_{mn} denotes the total weight that can be carried from m to n — the assumption being that it will be used in full.

Using the above two sets of equations as constraints we can now use the linear programming format to minimise an objective function defining total costs (labour and transportations) as follows :

$$\sum_m \sum_n \sum_i {}_{mn}X_i \left({}_mP_i + {}_{mn}P'_i\right) + \sum_m \sum_u \sum_i \left({}_mX_i - {}_{mn}X_i\right)_m P_i$$

V denotes production costs and P' donotes the cost of transportation from m to n. Other symbols are defined as before.

In the above set we now bring the capital-output ratios and the capital input ratios defined earlier. The balance equations are reformulated for the regions in terms of capital in productive use as follows :

$$\sum_j {}_n a_{ij}\ {}_n\lambda_j\ {}_nK_j + \sum_m {}_m\lambda_i \cdot {}_{nm}K_i - \sum_m {}_n\lambda_i \cdot {}_{nm}K_i + {}_nC_i$$
$$+ \sum_j {}_nb_{ij} \cdot \Delta K_j = {}_n\lambda_i \cdot {}_nK_i$$

Assumption has been made here that the output capital ratio X differs both in regions and industries.

The transportation constraint can be recast in capital terminology as below :

$$\Sigma\ W_j \cdot {}_{mn}\tau_{ij} \cdot X_j = {}_{mn}\tau \cdot \chi_{mn}$$

where x is capital employed to carry a volume of transport T and x is the ratio of capital to output. We thus have

$$T = \tau \cdot \chi \cdot$$

We now define the objective function as follows r

Minimise,

$$\sum_n \sum_j {}_nK_j + \sum_m \sum_n \sum_j {}_{mn}\chi_j \cdot$$

Here X and T represent the capital on production other than transportation and transportation activities respectively.

The objective function thus gives us an expression for the capital employed to produce in all the regions for all the sectors and the transportation capital employed to carry the goods of all types to the different destinations.

Minimising this objective function subject to the two sets of equations given before we come to the regional production and distribution pattern that makes the best use of available capital in the different regions and also fulfills the final demand targets.

It is to be noted here that the investment targets of the next period are arbitrarily fixed so that the optimisation procedure is strictly at one point of time. Investment allocation for a future period thus is not a subject of optimisation in the present model but the present employment of capital is.

This optimisation procedure thus gives regional flow pattern that may be used at present most effectively to produce the required final demands.

This is only one way in which the problem can be formulated for planning purposes. There may be others depending on the objectives of the planners but in many of them this simultaneous use of the capital output ratios or labour output ratio and input coefficients together, may lead to many quick procedures of efficient planning where considerations of capital or labour are important to the planners.